Renormalisation of noncommutative ϕ^4_4-theory to all orders

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$\longrightarrow$ **hep-th/**$\{$0305066, 0307017, 0401128, 0402093, 0403232$\}$

I. Introduction

– classical field theory is purely geometrical

– limits to localisability when combining principles of quantum mechanics and general relativity

$\Rightarrow$ space-time cannot have points and, therefore, is not a manifold

what else?

proposals: String Theory, Noncommutative Geometry, ...

• remarkable connection: zero-slope limit of string theory in presence of D-branes with Neveu-Schwarz B-field yields (super) Yang-Mills theory on a noncommutative geometry

• renormalisation?

– OK at one-loop

– fails at higher loop order due to UV/IR mixing

II. The ϕ^4-model on the 4D Moyal plane

- $\star$-product: $(a \star b)(x) := \int d^4y \frac{d^4k}{(2\pi)^4} \, a(x+\tfrac{1}{2}\theta\cdot k) b(x+y) \, \mathrm{e}^{iky}$
- lesson of UV/IR-mixing: *noncommutativity relevant at short distances modifies physics at very large distances*

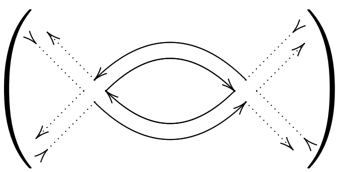
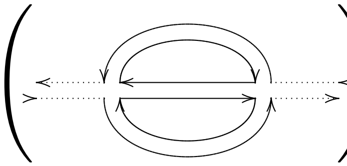
Theorem. The quantum field theory associated with the action

$$S = \int d^4x \Big(\frac{1}{2} \partial_\mu \phi \star \partial^\mu \phi + \frac{\Omega^2}{2} (\tilde{x}_\mu \phi) \star (\tilde{x}^\mu \phi) + \frac{\mu^2}{2} \phi \star \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \Big)(x) \;,$$

for $\tilde{x}_\mu := 2(\theta^{-1})_{\mu\nu} x^\nu$, ϕ-real, Euclidean metric, is perturbatively renormalisable to all orders in λ.

- Remark: the above action is covariant with respect to a duality between position space and momentum space [Langmann & Szabo, 2002]:

$p_\mu \leftrightarrow \tilde{x}_\mu \;, \qquad \hat{\phi}(p) \leftrightarrow \pi^2 \sqrt{|\det \theta|}\, \phi(x) \Rightarrow \qquad S[\phi; \mu_0, \lambda, \Omega] \mapsto \Omega^2 S[\phi; \frac{\mu_0}{\Omega}, \frac{\lambda}{\Omega^2}, \frac{1}{\Omega}]$

dual of $\Big($  $\Big)(p)$ divergent $\Rightarrow$ dual of $\Big($  $\Big)(p)$ divergent

III. Reformulation as a dynamical matrix model

- there is a base $\{b_{mn}(x)\}_{m,n\in\mathbb{N}^2}$ of the 4D-Moyal plane which fulfils

$$(b_{mn}\star b_{kl})(x) = \delta_{nk} b_{ml}(x), \qquad \int d^4x\, b_{mn}(x) = (2\pi\theta)^2\,\delta_{mn}$$

$\rightarrow$ non-local $\star$-product interaction becomes simple matrix product . . .

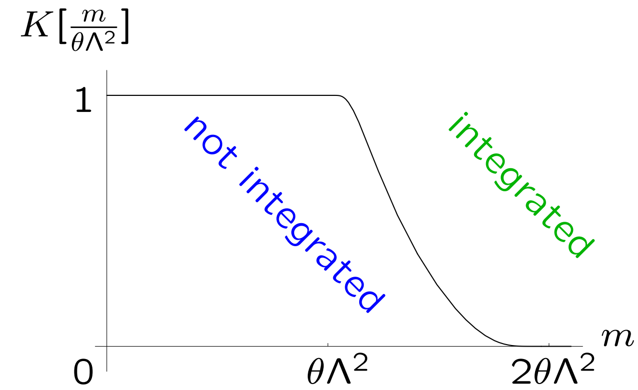
$$S = (2\pi\theta)^2 \sum_{m,n,k,l\in\mathbb{N}^2} \Big(\frac{1}{2}\phi_{mn} G_{mn;kl}\phi_{kl} + \frac{\lambda}{4!}\phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}\Big)$$

. . . but kinetic term $G_{mn;kl}$ and propagator $\Delta = G^{-1}$ are complicated:

$$\begin{aligned}
\Delta_{\substack{m^1 n^1; k^1 l^1\\ m^2 n^2; k^2 l^2}} &= \frac{\theta}{2(1+\Omega)^2}\delta_{m^1+k^1,n^1+l^1}\delta_{m^2+k^2,n^2+l^2} \\
&\times \sum_{v^1=\frac{|m^1-l^1|}{2}}^{\frac{m^1+l^1}{2}} \sum_{v^2=\frac{|m^2-l^2|}{2}}^{\frac{m^2+l^2}{2}} B\Big(1+\tfrac{\mu_0^2\theta}{8\Omega}+\tfrac{1}{2}(m^1+k^1+m^2+k^2)-v^1-v^2, 1+2v^1+2v^2\Big) \\
&\times {}_2F_1\Bigg(\begin{matrix}1+2v^1+2v^2\,,\ \frac{\mu_0^2\theta}{8\Omega}-\frac{1}{2}(m^1+k^1+m^2+k^2)+v^1+v^2\\ 2+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(m^1+k^1+m^2+k^2)+v^1+v^2\end{matrix}\Bigg|\frac{(1-\Omega)^2}{(1+\Omega)^2}\Bigg)\Big(\frac{1-\Omega}{1+\Omega}\Big)^{2v^1+2v^2} \\
&\times \prod_{i=1}^{2}\sqrt{\binom{n^i}{v^i+\frac{n^i-k^i}{2}}\binom{k^i}{v^i+\frac{k^i-n^i}{2}}\binom{m^i}{v^i+\frac{m^i-l^i}{2}}\binom{l^i}{v^i+\frac{l^i-m^i}{2}}}
\end{aligned}$$

IV. Renormalisation by flow equations

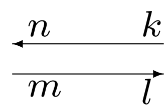
- integration of matrix modes above $\Lambda^2\theta$ in partition function leads to effective action $L[\phi,\Lambda]$

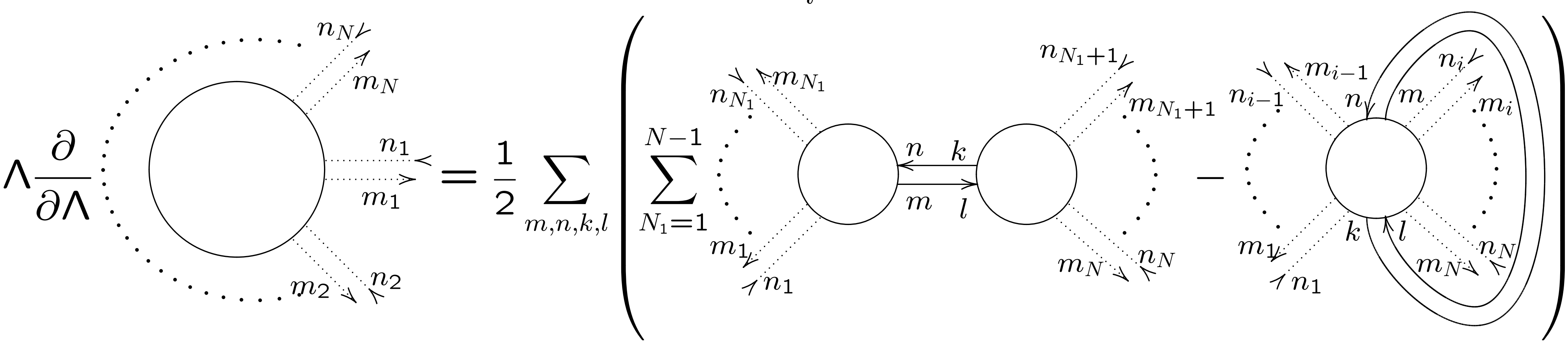


- renormalisation amounts to prove that the matrix Polchinski equation

$$\boxed{\Lambda\frac{\partial L[\phi,\Lambda]}{\partial\Lambda} = \sum_{m,n,k,l}\frac{1}{2}\underbrace{\Lambda\frac{\partial\Delta^K_{mn;kl}(\Lambda)}{\partial\Lambda}}_{Q_{mn;kl}}\Big(\frac{\partial L[\phi,\Lambda]}{\partial\phi_{mn}}\frac{\partial L[\phi,\Lambda]}{\partial\phi_{kl}} - \frac{1}{(2\pi\theta)^2}\frac{\partial^2 L[\phi,\Lambda]}{\partial\phi_{mn}\,\partial\phi_{kl}}\Big)}$$

admits a regular solution which depends on finitely many inital data

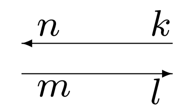
- graphical interpretation:
  ribbon graphs with propagator  $= Q_{mn;kl}(\Lambda)$

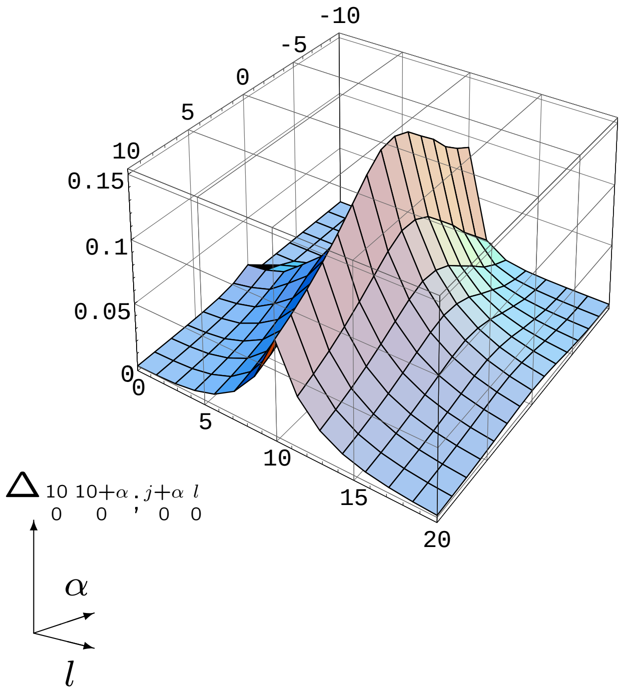


– drawn on Riemann surface of genus $g = 1 - \frac{1}{2}(L - I + V)$ with B holes

V. Power-counting theorem

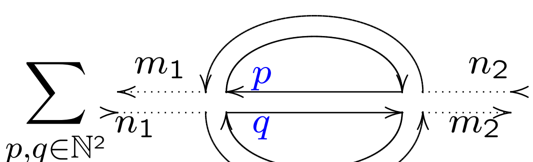
$$\left|L^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\right| \leq \left(\sqrt{\theta}\Lambda\right)^{(4-N)+4(1-B-2g)} \mathrm{pol}^{4V-N}\left[\frac{\{m_i^r,n_i^r\}}{\theta\Lambda^2}\right]\mathrm{pol}^{2V-\frac{N}{2}}\left[\ln\frac{\Lambda}{\Lambda_R}\right]$$

- explanation: properties of the propagator $Q_{mn;kl}(\Lambda) =$ 

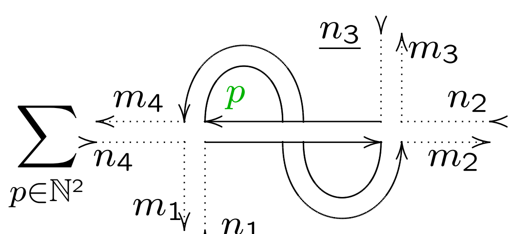


$$\left.\begin{array}{l}\max\limits_{m,n,k,l}|Q_{nm;lk}(\Lambda)| \leq \dfrac{C}{\theta\Lambda^2}\\ \text{index volume} \sim \theta^2\Lambda^4\end{array}\right\} \to \text{power-counting degree } 4-N$$

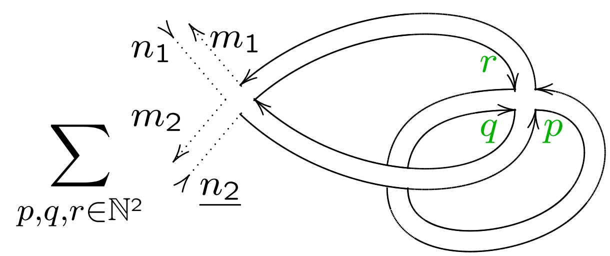
$$\left.\begin{array}{l}\Delta_{mn;kl} \text{ has for given } m\\ \text{sharp maximum at } l{=}m:\\ \displaystyle\sum_l\Big(\max_{n,k}|Q_{\underline{m}n;kl}(\Lambda)|\Big) \leq \frac{C'}{\theta\Lambda^2}\end{array}\right\} \to \text{no volume factor required}$$



$B=1, g=0$

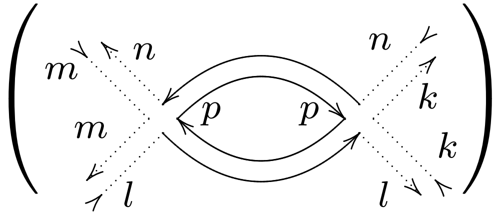


$B=2, g=0$



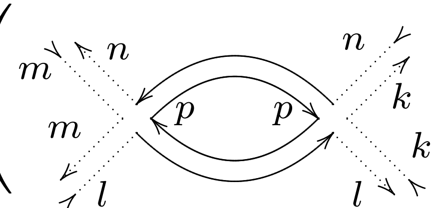
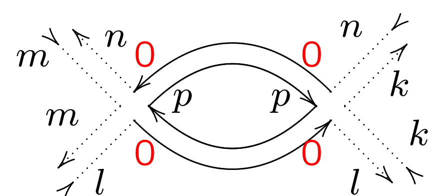
$B=1, g=1$

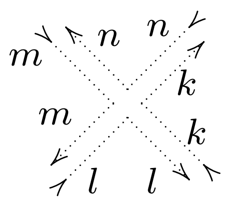
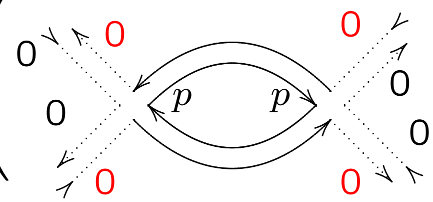
Integration procedure

- Polchinski equation: $\Lambda' \dfrac{\partial}{\partial \Lambda'} L^{(2,1,0)}_{mn;nk;kl;lm}[\Lambda'] = \sum_{p \in \mathbb{N}^2} \Big($  $\Big)(\Lambda')$

– requirement: achieve bounded $L[\lambda]$ by finitely many initial data

$\Rightarrow$ we need mixed boundary conditions for the integration

$$A^{(2,1,0)}_{mn;nk;kl;lm}[\Lambda] = -\int_{\Lambda}^{\Lambda_0} \frac{d\Lambda'}{\Lambda'} \sum_{p \in \mathbb{N}^2} \Big($$  $-$  $$\Big)(\Lambda')$$

$+$  $\Big[\int_{\Lambda_R}^{\Lambda} \frac{d\Lambda'}{\Lambda'} \sum_{p \in \mathbb{N}^2} \Big($  $\Big)(\Lambda') + A^{(2,1,0)}_{00;00;00;00}[\Lambda_R] \Big]$

- discrete Taylor expansion $\Rightarrow$ difference to reference graph irrelevant

- similar mixed boundary conditions required for

  $L^{(V,1,0)}_{\genfrac{}{}{0pt}{}{m^1}{m^2}\genfrac{}{}{0pt}{}{n^1}{n^2};\genfrac{}{}{0pt}{}{n^1}{n^2}\genfrac{}{}{0pt}{}{m^1}{m^2}}$ (2 conditions), $L^{(V,1,0)}_{\genfrac{}{}{0pt}{}{m^1+1}{m^2}\genfrac{}{}{0pt}{}{n^1+1}{n^2};\genfrac{}{}{0pt}{}{n^1}{n^2}\genfrac{}{}{0pt}{}{m^1}{m^2}}$, $L^{(V,1,0)}_{\genfrac{}{}{0pt}{}{m^2}{m^1}\genfrac{}{}{0pt}{}{n^2}{n^1};\genfrac{}{}{0pt}{}{n^2}{n^1}\genfrac{}{}{0pt}{}{k^2}{k^1};\genfrac{}{}{0pt}{}{k^2}{k^1}\genfrac{}{}{0pt}{}{l^2}{l^1};\genfrac{}{}{0pt}{}{l^2}{l^1}\genfrac{}{}{0pt}{}{m^2}{m^1}}$

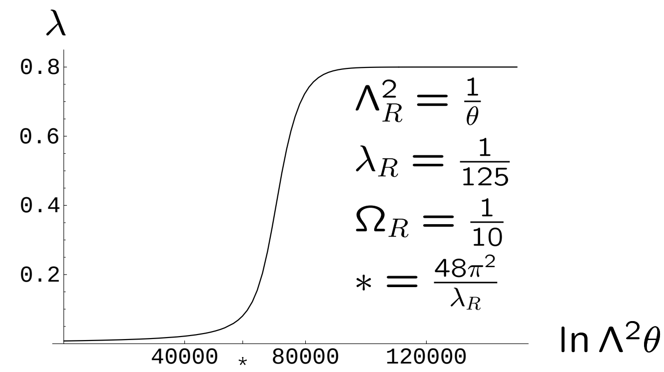
  (with identical index dependence as initial action)

$\Rightarrow$ model is renormalisable by normalisation conditions for mass, field amplitude, coupling constant and oscillator frequency

VI. Outlook

- one-loop β-function:

$$\frac{\lambda[\Lambda]}{\Omega^2[\Lambda]} = \text{const}$$



running coupling constant can be kept small

$\Rightarrow$ nonperturbative treatment is possible (joint work with V. Rivasseau)

- most important project: gauge theories

$\rightarrow$ need gauge-invariant action with oscillator potential

– via Dirac operator with oscillator potential and spectral action?
– via covariant coordinates?
– hints from string theory?

- general observation:

UV/IR-duality in nc quantum field theories induces a compactification (seems unphysical, but there are cosmological hints in WMAP data)

$\rightarrow$ if Nature abhors the infinitesimal small (strings, noncommutativity), it cannot permit the infinitely large. . .